SCUOLA SUPERIORE UNIVERSITARIA DI TOPPO-WASSERMANN

Middle year thesis

Model the uncertainty in a logical way

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Accademic year 2023/2024

Introduction

How to formalize knowledge was one of the first questions raised by philosophers. The main tool used so far to this aim is logic: in philosophy and mathematics, there is no doubt that a deep understanding and a correct use of it prevent us from some fallacious or vicious arguments.

In recent times, logical questions have become central in another field of study: informatics. We should think not only about computability theory or programming languages because logic is also used to control, verify, and guarantee that the algorithms that we design will work as we expect: this is the goal of program verification.

The role of logic is even more central if we think about artificial intelligence. The golden goal of AI researchers is to build a machine whose reasoning abilities will overcome humans and we are moving closer and closer to this revolutionary moment: think about ChatGPT, Bard, AlphaGo, or AlphaGeometry. Another aim that the most ambitious are pursuing is the design of a *artificial general intelligence (AGI)*: an agent that not only astonishingly performs a precise task (like playing Go for AlphaGo or solving mathematical problems for AlphaGeometry) but one whose area of expertise is wider, as a human being; an agent that can understand the surrounding world, that can decide what to observe and what to ask himself, that can take actions and influence other agents' behavior.

What should we expect from the existence of such intelligent agents? Should we look forward to an AGI or should the thrill of playing with something (or someone?) over which we can't have any control frighten us? Despite the answer that everyone could give to these questions, we think that it would be very reassuring, even for the enthusiastic supporter, to have a way that protects us from an eventual maleficient agent of this kind. This is the aim of AGI safety, an interdisciplinary field of study that is aimed at developing methods to ensure that these powerful agents will behave in a human safe way.

This work finds its reason in some of the research agendas of this field. Formalizing reasoning is one of the first ways that comes to mind to guarantee that an agent will behave safely. But it seems that logic alone doesn't fit in for this task: indeed, we need also some tools for dealing with uncertainty, in both the observing stage of the agent and the information processing one. Some authors, then, thought about combining logic and probability: the situation that we have in mind is an agent that tries to infer some information from the collected/observed datas, by assigning a certain degree of belief to formulas. This will be formalized taking "probabilities" defined on the set of sentences of a given language, as it is shown in this work: the value assigned to a sentence should be interpreted as "how much truth" we will recognize in the statement.

We hope that even a reader who is not interested in AGI safety or regards the design of an AGI as a remote event will find the topic fascinating both in a mathematical sense and in a philosophical one.

Contents

Chapter 1

Probabilities on sentences

In the following, we will consider languages $\mathcal{L}$ that are countable, i.e. with a countable supply of variables ($x_0, \dots, x_n, \dots$ or $y_0, \dots, y_n, \dots$ or simply x, y, z) and that can have at most a countable supply of constants ($c_0, \dots, c_n$) predicate symbols ($A_0, \dots A_n, \dots$) and of function symbols ($f_0, \dots, f_n, \dots$ or simply $f, g, h, \dots$). When the countability hypothesis plays a central role, we will stress it, however. The logic that we will use is the first-order classical one and the set of all sentences that can be built with the boolean connectives ($\wedge, \vee, \neg, \rightarrow, \equiv$) and the first-order quantifiers ($\exists, \forall$) with the non-logical symbols of $\mathcal{L}$ will be denoted with $\mathrm{Sen}(\mathcal{L})$.

The first intuition we have about probability on sentences will be formalized by the following definition.

Definition 1.0.1. Given a language $\mathcal{L}$, a *probability* on $\mathcal{L}$ is a map $\mathbb{P} : \mathrm{Sen}(\mathcal{L}) \to \mathbb{R}_{\geq 0}$ such that:

(a) for every valid φ, $\mathbb{P}(\varphi) = 1$;

(b) if $\neg(\varphi \wedge \psi)$ is valid, then $\mathbb{P}(\varphi \vee \psi) = \mathbb{P}(\varphi) + \mathbb{P}(\psi)$.

Example 1.0.1. We can consider a language $\mathcal{L}$ and a $\mathcal{L}$-model $\mathfrak{A}$. We can define the probability $\mathbb{P}_{\mathfrak{A}}$ that assigns 1 to all the sentences true in $\mathfrak{A}$ and 0 otherwise. This is a probability since:

- if φ is valid, then it is true in $\mathfrak{A}$ and, therefore, $\mathbb{P}_{\mathfrak{A}}(\varphi) = 1$;
- let φ and ψ be in contradiction. Then $\mathfrak{A} \vDash \neg(\varphi \wedge \psi)$ and thus in $\mathfrak{A}$ only one among φ and ψ can be true. We have two cases:
  - if $\varphi \vee \psi$ is true in $\mathfrak{A}$, then $\mathbb{P}_{\mathfrak{A}}(\varphi \vee \psi) = 1$. Furthermore, at least one among φ and ψ is true in $\mathfrak{A}$ and, by the previous remark, exactly one of them is, from which the condition b) of Definition 1.0.1 is satisfied;

- otherwise, $\varphi \vee \psi$ is not true in $\mathfrak{A}$ and so is φ and ψ. Therefore, $\mathbb{P}_{\mathfrak{A}}(\varphi \vee \psi) = \mathbb{P}_{\mathfrak{A}}(\varphi) = \mathbb{P}_{\mathfrak{A}}(\psi) = 0$, hence the thesis.

Actually, the just defined $\mathbb{P}_{\mathfrak{A}}$, is a truth-values valuation: therefore, this example shows that all these valuations are nothing but probabilities on $\mathrm{Sen}(\mathcal{L})$.

Example 1.0.2. Another example of probability is given by taking linear combinations of the probabilities in Example 1.0.1: given some $\mathcal{L}$-structures $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ and some reals $m_1, \ldots, m_n$ that sum to 1, we have also the probability

$$\mathbb{P} = \sum_{i=1}^{n} m_i \, \mathbb{P}_i \, .$$

We can generalize the example above in the case of a countable quantity of structures: we can choose some models $\{\mathfrak{A}_i\}_{i \in \mathbb{N}}$ and some weights $\{m_i\}_{i \in \mathbb{N}}$ such that $\sum_{i \in \mathbb{N}} m_i = 1$ and define a new probability $\mathbb{P} = \sum_{i \in \mathbb{N}} m_i \, \mathbb{P}_{\mathfrak{A}_i}$. Specific examples of this form will be used in Theorem 1.3.1 and in Theorem 1.3.2.

Perhaps, the reader has already encountered such examples of probabilities in logics expressivity results. For instance, suppose that $\mathcal{L}$ is a finite language without functional symbols. Since[1], for fixed $n \in \mathbb{N}_{>0}$, there are a finite number of structures of cardinality n up to isomorphism, we can define for all $n > 0$ the (finite) set $\mathcal{C}_n$ of all the $\mathcal{L}$-structures with n elements in the domain. Therefore, for every formula φ, let be

$$\mathbb{P}_n(\varphi) := \frac{|\{\mathfrak{A} \in \mathcal{C}_n : \mathfrak{A} \vDash \varphi\}|}{|\mathcal{C}_n|}.$$

These $\mathbb{P}_n$'s are probabilities in the sense of Definition 1.0.1 and they are actually linear combinations (with all the weights equal to $\frac{1}{|\mathcal{C}_n|}$) of probabilities of the form $\mathbb{P}_{\mathfrak{A}}$ for an $\mathcal{L}$-structure $\mathfrak{A}$.

The limit as n approaches to infinity of these probabilities may exist and in this case it is also a probability. The $0 - 1$ Law, for first-order logic, states that for every formula $\varphi \in \mathrm{Sen}(\mathcal{L})$, the limit exists (usually it is called *asymptotic probability*) and the possible outcomes are only 0 or 1.

The conditions above are sufficient to guarantee other properties of a probability, as the Proposition 1.0.1 shows.

[1]The absence of functional symbols is necessary for the proof of the theorem we will discuss later in the example; we don't need this assumption to have only finitely many not isomorphic structures with a given cardinality.

Proposition 1.0.1. *Let $\mathbb{P}$ be a probability over a language $\mathcal{L}$. Then for every φ, ψ and $\varphi_i \in \mathrm{Sen}(\mathcal{L})$ the following hold:*

1) $\mathbb{P}(\neg\varphi) = 1 - \mathbb{P}(\varphi)$*;*

2) $\mathbb{P}(\varphi) \in [0, 1]$*;*

3) if φ is unsatisfiable, then $\mathbb{P}(\varphi) = 0$;

4) if $\varphi \to \psi$ is valid, then $\mathbb{P}(\varphi) \leq \mathbb{P}(\psi)$;

5) if $\varphi \equiv \psi$ is valid, then $\mathbb{P}(\varphi) = \mathbb{P}(\psi)$;

6) $\mathbb{P}(\varphi \wedge \psi) \leq \mathbb{P}(\psi)$ and the equality holds if φ is valid;

7) $\mathbb{P}(\bigvee_{i=1}^{n} \varphi_i) \leq \sum_{i=1}^{n} \mathbb{P}(\varphi_i)$ and the two terms are equal if the sentences $\varphi_1, \ldots, \varphi_n$ are pairwise contradictory sentences (i.e. for every distinct $i, j \in \{1, \ldots, n\}$, $\varphi_i \wedge \varphi_j$ is unsatisfiable);

8) $\mathbb{P}(\varphi \vee \psi) + \mathbb{P}(\varphi \wedge \psi) = \mathbb{P}(\varphi) + \mathbb{P}(\psi)$.

9) if $\mathbb{P}(\psi) > 0$, the map $\mathbb{P}(-|\psi) : \mathrm{Sen}\,\mathcal{L} \to \mathbb{R}_{\geq 0}$ defined by

$$\mathbb{P}(\varphi|\psi) = \frac{\mathbb{P}(\varphi \wedge \psi)}{\mathbb{P}(\psi)},$$

is a probability.

Proof. 1) Since $\neg(\varphi \wedge \neg\varphi)$ is valid, then $\mathbb{P}(\varphi \vee \neg\varphi) = \mathbb{P}(\varphi) + \mathbb{P}(\neg\varphi)$. The first term of the precedent equality is 1 because $\varphi \vee \neg\varphi$ is valid, then the thesis;

2) By Definition 1.0.1, $\mathbb{P}(\varphi) \geq 0$ but also $\mathbb{P}(\neg\varphi) \geq 0$. Thanks to 1), we have $\mathbb{P}(\varphi) = 1 - \mathbb{P}(\neg\varphi) \leq 1$;

3) It follows from 1) and the fact that the negation of an unsatisfiable formula is a valid one;

4) If $\varphi \to \psi$ is valid, then so $\psi \vee \neg\varphi$ is and therefore $\neg(\neg\psi \wedge \varphi)$ is. By b) of Definition 1.0.1,

$$\mathbb{P}(\neg\psi \vee \varphi) = \mathbb{P}(\neg\psi) + \mathbb{P}(\varphi)$$

and therefore, using 1) and 2)

$$1 \geq \mathbb{P}(\neg\psi \vee \varphi) = \mathbb{P}(\neg\psi) + \mathbb{P}(\varphi) = 1 - \mathbb{P}(\psi) + \mathbb{P}(\varphi),$$

from which the thesis.

5) It follows from 4)

6) By the excluded middle principle, ψ is equivalent to $(\psi \wedge \varphi) \vee (\psi \wedge \neg\varphi)$. Therefore by 5) and by the fact that $\psi \wedge \varphi$ is in contradiction with $\psi \wedge \neg\varphi$, we have

$$\mathbb{P}(\psi) = \mathbb{P}((\psi \wedge \varphi) \vee (\psi \wedge \neg\varphi)) = \mathbb{P}(\psi \wedge \varphi) + \mathbb{P}(\psi \wedge \neg\varphi) \geq \mathbb{P}(\psi \wedge \varphi).$$

In the case in which φ is valid, then $\psi \wedge \neg\varphi$ is unsatisfiable, and by 3) the last disequality in the expression above is indeed an equality.

7) Assume first that $\varphi_1, \ldots, \varphi_n$ are pairwise contradictory. Then, also $\varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_{n-1}$ and φ_n are in contradiction and by b) of Definition 1.0.1,

$$\mathbb{P}(\varphi_1 \vee \cdots \vee \varphi_n) = \mathbb{P}((\varphi_1 \vee \cdots \vee \varphi_{n-1}) \vee \varphi_n) = \mathbb{P}(\varphi_1 \vee \cdots \vee \varphi_{n-1}) + \mathbb{P}(\varphi_n).$$

Then, it is easy to see that the thesis in the case of pairwise contradictory sentences follows by induction.

In the general case, we can notice that

$$\bigvee_{i=1}^{n} \varphi_i \equiv \bigvee_{i=1}^{n-1} \varphi_i \vee (\neg \bigvee_{i=1}^{n-1} \varphi_i \wedge \varphi_n)$$

and that $\bigvee_{i=1}^{n-1} \varphi_i$ and $(\neg \bigvee_{i=1}^{n-1} \varphi_i \wedge \varphi_n)$ are in contradiction. Therefore by 5), b) of Definition 1.0.1 and 6), we have

$$\begin{aligned}\mathbb{P}(\bigvee_{i=1}^{n} \varphi_i) = \mathbb{P}(\bigvee_{i=1}^{n-1} \varphi_i \vee (\neg \bigvee_{i=1}^{n-1} \varphi_i \wedge \varphi_n)) &= \mathbb{P}(\bigvee_{i=1}^{n-1} \varphi_i) + \mathbb{P}(\neg \bigvee_{i=1}^{n-1} \varphi_i \wedge \varphi_n)) \\ &\leq \mathbb{P}(\bigvee_{i=1}^{n-1} \varphi_i) + \mathbb{P}(\varphi_n).\end{aligned}$$

As in the previous case, it's easy to see now that a simple induction yields to the thesis.

8) We can notice that $\varphi \vee \psi$ is equivalent to the formula $\varphi \vee (\neg\varphi \wedge \psi)$, that has the two disjuncts in contradiction. Therefore

$$\mathbb{P}(\varphi \vee \psi) = \mathbb{P}(\varphi \vee (\neg\varphi \wedge \psi)) = \mathbb{P}(\varphi) + \mathbb{P}(\neg\varphi \wedge \psi). \tag{1.1}$$

Using that ψ is equivalent to $(\varphi \wedge \psi) \vee (\neg\varphi \wedge \psi)$ and that the disjuncts on this formula are in contradiction, we get

$$\mathbb{P}(\psi) = \mathbb{P}(\varphi \wedge \psi) + \mathbb{P}(\neg\varphi \wedge \psi). \tag{1.2}$$

Combining Equation (1.1) and Equation (1.2) we have the thesis.

9) The condition a) of Definition 1.0.1 holds by 6). To show the validity of the condition b), let φ_1 and φ_2 two sentences that are in contradiction. Logically $(\varphi_1 \vee \varphi_2) \wedge \psi$ is equivalent to $(\varphi_1 \wedge \psi) \vee (\varphi_2 \wedge \psi)$ and the two disjuncts are in contradiction by the hypothesis we are assuming about φ_1 and φ_2. Therefore

$$\mathbb{P}((\varphi_1 \vee \varphi_2) \wedge \psi) = \mathbb{P}(\varphi_1 \wedge \psi) + \mathbb{P}(\varphi_2 \wedge \psi),$$

and then

$$\begin{aligned} \mathbb{P}(\varphi_1 \vee \varphi_2|\psi) &= \frac{\mathbb{P}((\varphi_1 \vee \varphi_2) \wedge \psi)}{\mathbb{P}(\psi)} = \frac{\mathbb{P}(\varphi_1 \wedge \psi) + \mathbb{P}(\varphi_2 \wedge \psi)}{\mathbb{P}(\psi)} \\ &= \mathbb{P}(\varphi_1|\psi) + \mathbb{P}(\varphi_2|\psi). \end{aligned}$$

□

Remark 1.0.1. The condition 3) of Proposition 1.0.1 can't be inverted. Indeed, it's possible to find a probability $\mathbb{P}$ and a satisfiable sentence φ such that $\mathbb{P}(\varphi) = 0$. For instance, if we take the language $\mathcal{L} = \{0, S\}$ where 0 is a constant and S a unary function symbol, we have a model $\mathfrak{A} = (A, 0^{\mathfrak{A}}, S^{\mathfrak{A}})$ defined by the following:

- the domain A of the model is the set $\{a, b\}$;
- $0^{\mathfrak{A}}$ is the element a;
- $S^{\mathfrak{A}}$ is the function that maps all the elements of A in a.

With the notation of Example 1.0.1, we can define the probability $\mathbb{P}_{\mathfrak{A}}$. For every formula φ that is satisfiable but not true in $\mathfrak{A}$ (for instance, the formula "$\forall x S(x) \neq 0$" that is true in the standard model of arithmetic) we have $\mathbb{P}_{\mathfrak{A}}(\varphi) = 0$, thus an example of a satisfiable statement with assigned a null probability.

Remark 1.0.2. The condition 7) of Proposition 1.0.1 is not an equivalence: there can be a probability $\mathbb{P}$ on $\mathrm{Sen}(\mathcal{L})$ and two non contradictory statements φ_1, φ_2 with $\mathbb{P}(\varphi_1 \vee \varphi_2) = \mathbb{P}(\varphi_1) + \mathbb{P}(\varphi_2)$. As an example, we can take the probability $\mathbb{P}_{\mathfrak{A}}$ and the formula φ that we used in Remark 1.0.1. If we define φ_1 and φ_2 to be φ, then these two formulas are not in contradiction and $\mathbb{P}_{\mathfrak{A}}$ assigns 0 to $\mathbb{P}(\varphi_1 \vee \varphi_2), \mathbb{P}(\varphi_1)$ and $\mathbb{P}(\varphi_2)$, guaranteeing the validity of the required equation.

Thanks to 9) of Proposition 1.0.1, we can define, whenever $\mathbb{P}(\psi) > 0$, the *conditional probability* $\mathbb{P}(-|\psi)$ as pointed out above.

Furthermore, Proposition 1.0.1 explains to us why the maps in Definition 1.0.1 are called probabilities and not simply measures: indeed, they can be regarded as maps from $\mathrm{Sen}(\mathcal{L})$ to $[0,1]$ and not to all $\mathbb{R}_{\geq 0}$. Even if some similarities with the usual notion of "probability"can be detected, the reader with a more measure-theoretic approach will however have some trouble with the use of this term for such functions: the following section is devoted to the connection between these, for now different, concepts.

1.1 Measure-theoretic approach

In measure theory, to regard a map as a probability, we need some kind of structure in the domain of this function.

Definition 1.1.1. Let Ω be a non-empty set and $\mathcal{C} \subseteq \mathcal{P}(\Omega)$.

- $\mathcal{C}$ is an algebra if it is closed under all the finite boolean combinations (finite union, finite intersection, and complementation)[2];
- $\mathcal{C}$ is a σ-algebra if it is closed under all the finite and countable boolean combinations.

If $\mathcal{C}$ is a σ-algebra [algebra] on Ω, a *probability* on $\mathcal{C}$ is a map $\mathbb{P} : \mathcal{C} \to [0,1]$ such that:

- is non trivial, i.e. $\mathbb{P}(\emptyset) \neq \mathbb{P}(\Omega)$;
- is finitely additive, i.e. for all $n \in \mathbb{N}$ and for all disjoint elements $A_1, \ldots, A_n$ in $\mathcal{C}$,
$$\mathbb{P}(\dot{\bigcup}_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mathbb{P}(A_i)$$
- is [conditionally] σ-additive, i.e. if $\{A_i\}_{i\in\mathbb{N}}$ is a countable set of disjoint elements in $\mathcal{C}$ [and if the union $\dot{\bigcup}_{i\in\mathbb{N}} A_i$ belongs to $\mathcal{C}$], then $\mathbb{P}(\dot{\bigcup}_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} \mathbb{P}(A_i)$.

From this definition follow some basic properties of a probability that we don't recall here in detail. For example, the fact that $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$ and that the finitely additive property is equivalent to the request that for any disjoint $A, B \in \mathcal{C}$, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

[2] We are including the empty union (i.e. $\emptyset$) and the empty intersection (i.e. the entire Ω)

In the case of a probability $\mathbb{P}$ defined on an algebra/σ-algebra $\mathcal{C}$ on Ω, we will usually say that $\mathbb{P}$ is a probability on Ω, when $\mathcal{C}$ is obvious from the context.

The conditionally σ-additive condition is sufficient also to guarantee the finitely additive one. However, we decided to explicitly point out this feature in order to understand better the following proposition that we will use later.

Proposition 1.1.1. *Let $\mathcal{C}$ be an algebra on Ω and $\mathbb{P}$ a non trivial and finitely additive map $\mathcal{C} \to [0,1]$. The following are equivalent:*

- *$\mathbb{P}$ is a probability, i.e. $\mathbb{P}$ is also conditionally σ-additive;*
- *for every family $\{A_i\}_{i\in\mathbb{N}}$ of elements in $\mathcal{C}$ such that $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$ and that $\bigcup_{i\in\mathbb{N}} A_i$ is an element in $\mathcal{C}$,*
$$\mathbb{P}(\bigcup_{i\in\mathbb{N}} A_i) = \lim_{i\to+\infty} \mathbb{P}(A_i);$$
- *$\mathbb{P}$ is σ-subadditive, i.e. for every family $\{A_i\}_{i\in\mathbb{N}}$ of elements in $\mathcal{C}$ such that $\bigcup_{i\in\mathbb{N}} A_i$ is an element of $\mathcal{C}$,*
$$\sum_{i\in\mathbb{N}} \mathbb{P}(A_i) \geq \mathbb{P}(\bigcup_{i\in\mathbb{N}} A_i).$$

The aim of the following is to understand whether there is a link between the notion of probability on sentences and the one defined in the measure-theoretic setting. In order to do so, we need first to detect a set Ω and an algebra/σ-algebra in the logical context.

The first idea that comes to mind is to consider the set $\mathrm{Mod}(\mathcal{L})$ of all models for a language $\mathcal{L}$. It is known that this set can be endowed with a topological structure $\mathcal{T}$, defining as a basis

$$\mathcal{B} = \{A \subseteq \mathrm{Mod}(\mathcal{L}) : \exists\varphi \in \mathrm{Sen}(\mathcal{L}) \quad A = \mathrm{Mod}(\varphi)\},$$

where $\mathrm{Mod}(\varphi)$ is the set of all the models in which φ holds: in this way, an open set is a (countable, since $\mathcal{L}$ is countable) union of elements in $\mathcal{B}$ and so we have a practical description of the elements in the topology $\mathcal{T}$. Actually, the elements in $\mathcal{B}$ are not only open but also closed sets of the topology: this is because $\mathrm{Mod}(\varphi)^c = \mathrm{Mod}(\neg\varphi)$.

Furthermore, this topology is compact and this will be of very use in the following theorems. A way to prove the compactness is to show that if we have a family of closed $\{C_i\}_i \in I$ such that $\bigcap_{i\in I} C_i = \emptyset$, then exists a finite subset F of I such that $\bigcap_{i\in F} C_i = \emptyset$. Two simple remarks get this proof easier: in primis, a closed set is the complement of an open set, hence it is a countable intersection of sets of the form $\mathrm{Mod}(\varphi)^c = \mathrm{Mod}(\neg\varphi)$ and then any intersection of closed sets is a countable intersection of sets in $\mathcal{B}$; in secundis, since $\mathcal{B}$ is countable, we have only to show that if $\bigcap_{i=0}^{\infty} \mathrm{Mod}(\varphi_i) = \emptyset$, then exists n such that $\bigcap_{i=0}^{n} \mathrm{Mod}(\varphi_i) = \emptyset$, for every set $\{\varphi_i\}_{i\in\mathbb{N}}$ of sentences of $\mathcal{L}$. This is actually the standard form of the Compactness Theorem: if the set $\{\varphi_i\}_{i\in\mathbb{N}}$ is not satisfiable, then it is not finitely satisfiable.

As it's very common to do, denoting in general with $\mathcal{F}(A)$ the smallest σ-algebra that contains the elements in A, we can define from $\mathcal{T}$ the borelian σ-algebra $\mathcal{F}(\mathcal{T})$.

Actually it can be shown, using that $\mathcal{L}$ is countable, that $\mathcal{F}(\mathcal{T}) = \mathcal{F}(\mathcal{B})$. Indeed, since $\mathcal{B} \subseteq \mathcal{T}$, surely $\mathcal{F}(\mathcal{T})$ is a σ-algebra that contains $\mathcal{B}$; furthermore, it is the smallest one, because if we have a σ-algebra $\mathcal{C}$ such that $\mathcal{F}(\mathcal{B}) \subseteq \mathcal{C}$, then $\mathcal{C}$ contains also $\mathcal{T}$ because it contains all the countable union of elements in $\mathcal{B}$ and every open set in $\mathcal{T}$ is such an union (here the countability of $\mathcal{L}$, and hence of $\mathcal{B}$ is used).

The last remark will be useful in the following because $\mathcal{B}$ has a nice measure theoretic property: it is an algebra. Indeed:

- it is closed under finite intersection since

$$\mathrm{Mod}(\mathcal{L}) = \mathrm{Mod}(\varphi \to \varphi)$$
$$\mathrm{Mod}(\varphi_1) \cap \cdots \cap \mathrm{Mod}(\varphi_n) = \mathrm{Mod}(\varphi_1 \wedge \cdots \wedge \varphi_n);$$

- it is closed under finite union since

$$\emptyset = \mathrm{Mod}(\varphi \wedge \neg\varphi)$$
$$\mathrm{Mod}(\varphi_1) \cup \cdots \cup \mathrm{Mod}(\varphi_n) = \mathrm{Mod}(\varphi_1 \vee \cdots \vee \varphi_n);$$

- it is closed under complementation since

$$\mathrm{Mod}(\varphi)^c = \mathrm{Mod}(\neg\varphi).$$

Hence, to sum up, we have $\mathrm{Mod}(\mathcal{L})$, an algebra $\mathcal{B}$ on it and the σ-algebra $\mathcal{F}(\mathcal{T}) = \mathcal{F}(\mathcal{B})$ generated by this: in the following, the latter will be called the $\mathrm{Mod}(\mathcal{L})$ borelian σ-algebra. The convenience of having described the latter σ-algebra as generated by an algebra (and not only by a topology) is clear in the statement of the following theorem attributed to Caratheodory, Hahn, and Kolmogorov.

Theorem 1.1.1 (Extension Theorem). *Let $\mathcal{C}$ be an algebra on Ω, $\mathcal{F}(\mathcal{C})$ the σ-algebra generated and $\mathbb{P}$ a probability defined on $\mathcal{C}$. Then $\mathbb{P}$ can be extended in a unique way to a probability on $\mathcal{F}(\mathcal{C})$.*

Now we are ready to establish the correspondence we talked about at the beginning of this section. In the following, by a probability on $\mathrm{Mod}(\mathcal{L})$, we mean a probability on the borel σ-algebra $\mathcal{F}(\mathcal{B})$.

Theorem 1.1.2. *Let $\mathbb{P}$ be a probability as in Definition 1.0.1 on $\mathrm{Sen}(\mathcal{L})$. Then, there is a probability $\mathbb{P}^*$ as in Definition 1.1.1 on $\mathrm{Mod}(\mathcal{L})$ such that for every $\varphi \in \mathrm{Sen}(\mathcal{L})$,*

$$\mathbb{P}(\varphi) = \mathbb{P}^*(\mathrm{Mod}(\varphi)).$$

Conversely, let $\mathbb{P}$ be a probability as in Definition 1.1.1 on $\mathrm{Mod}(\mathcal{L})$. Then there's a probability $\bar{\mathbb{P}}$ as in Definition 1.0.1 on $\mathrm{Sen}(\mathcal{L})$ such that for every $\varphi \in \mathrm{Sen}(\mathcal{L})$,

$$\bar{\mathbb{P}}(\varphi) = \mathbb{P}(\mathrm{Mod}(\varphi)).$$

Proof. Assume that $\mathbb{P}$ is a probability on $\mathrm{Sen}(\mathcal{L})$. Then we can define a map $p^* : \mathcal{B} \to [0, 1]$ such that

$$p^*(\mathrm{Mod}(\varphi)) = \mathbb{P}(\varphi).$$

Therefore:

- the domain of p^* is an algebra;
- p^* is non trivial (since $\mathbb{P}$ satisfies condition a) of Definition 1.0.1 and 3) of Proposition 1.0.1)
- p^* is finitely additive: indeed, if $\mathrm{Mod}(\varphi)$ and $\mathrm{Mod}(\psi)$ are disjoint, then φ and ψ are in contradiction. Therefore

  $$\begin{aligned} p^*(\mathrm{Mod}(\varphi) \,\dot\cup\, \mathrm{Mod}(\psi)) &= p^*(\mathrm{Mod}(\varphi \vee \psi)) = \mathbb{P}(\varphi \vee \psi) \\ &= \mathbb{P}(\varphi) + \mathbb{P}(\psi) = p^*(\mathrm{Mod}(\varphi)) + p^*(\mathrm{Mod}(\psi)) \end{aligned}$$

- p^* is σ-subadditive: indeed, if we have $\{\varphi_i\}_{i\in\mathbb{N}}$, such that $\bigcup_{i\in\mathbb{N}} \mathrm{Mod}(\varphi_i)$ is an element of $\mathcal{B}$, then exists φ such that $\bigcup_{i\in\mathbb{N}} \mathrm{Mod}(\varphi_i) = \mathrm{Mod}(\varphi)$. By compactness, since $\bigcup_{i\in\mathbb{N}} \mathrm{Mod}(\varphi_i) \,\dot\cup\, \mathrm{Mod}(\neg\varphi) = \mathrm{Mod}(\mathcal{L})$, we have n such that $\bigcup_{i=1}^n \mathrm{Mod}(\varphi_i) \dot\cup \mathrm{Mod}(\neg\varphi) = \mathrm{Mod}(\mathcal{L})$ and therefore $\bigcup_{i=1}^n \mathrm{Mod}(\varphi_i) = \mathrm{Mod}(\varphi)$.

  Then by 7) of Proposition 1.0.1, we have

  $$\begin{aligned} p^*(\mathrm{Mod}(\varphi)) &= p^*(\bigcup_{i=1}^n \mathrm{Mod}(\varphi_i)) = \mathbb{P}(\bigvee_{i=1}^n \varphi_i) \\ &\leq \sum_{i=1}^n \mathbb{P}(\varphi_i) \leq \sum_{i=1}^\infty \mathbb{P}(\varphi_i) = \sum_{i=1}^\infty p^*(\mathrm{Mod}(\varphi_i)). \end{aligned}$$

Therefore, we can apply Proposition 1.1.1 to show that p^* is a probability (in the sense of Definition 1.1.1) on $\mathcal{B}$. By Theorem 1.1.1, it can be extended to a probability $\mathbb{P}^*$ on $\mathrm{Mod}(\mathcal{L})$ and, by the way, p^* was defined, $\mathbb{P}^*$ satisfies the required condition.

Let's now assume that we have a probability $\mathbb{P}$ on the borelian σ-algebra of $\mathrm{Mod}(\mathcal{L})$. We can define $\bar{\mathbb{P}} : \mathrm{Sen}(\mathcal{L}) \to [0,1]$ such that $\bar{\mathbb{P}}(\varphi) = \mathbb{P}(\mathrm{Mod}(\varphi))$. This is a probability in the sense of Definition 1.0.1. Indeed:

a) if φ is valid, then $\mathrm{Mod}(\varphi) = \mathrm{Mod}(\mathcal{L})$ and therefore

$$\bar{\mathbb{P}}(\varphi) = \bar{\mathbb{P}}(\mathrm{Mod}(\varphi)) = \mathbb{P}(\mathrm{Mod}(\mathcal{L})) = 1.$$

b) if φ and ψ are contradictory, then $\mathrm{Mod}(\varphi) \cap \mathrm{Mod}(\psi) = \emptyset$. Thus,

$$\begin{aligned}\bar{\mathbb{P}}(\varphi \vee \psi) &= \mathbb{P}(\mathrm{Mod}(\varphi \vee \psi)) = \mathbb{P}(\mathrm{Mod}(\varphi) \,\dot{\cup}\, \mathrm{Mod}(\psi)) \\ &= \mathbb{P}(\mathrm{Mod}(\varphi)) + \mathbb{P}(\mathrm{Mod}(\psi)) = \bar{\mathbb{P}}(\varphi) + \bar{\mathbb{P}}(\psi)\end{aligned}$$

Hence, $\bar{\mathbb{P}}$ is actually a probability on the $\mathcal{L}$-sentences. □

1.2 Gaifman condition

Even if in the Definition 1.0.1 only a few properties mark a probability out, we showed in Proposition 1.0.1 that they are sufficient to prove some of the features we want for such maps. By now, however, we don't have any control over the formula with quantifiers; the following definition tries to point out a meaningful property in this sense. In the following $\varphi\{x/t_i\}$ will be denoted also by $\varphi(t_i)$ when the variable which we are substituting is clear from the context or irrelevant.

Definition 1.2.1. Let $\mathcal{L}$ a countable language and $\{t_i\}_{i\in\mathbb{N}}$ an enumeration[3] of all closed terms in $\mathcal{L}$. A probability $\mathbb{P}$ on $\mathrm{Sen}(\mathcal{L})$ satisfies the *Gaifman condition* (we say that $\mathbb{P}$ is *Gaifman*) if for every φ with one and only one free variable x, we have:

$$\mathbb{P}(\exists x\varphi) = \sup\{\mathbb{P}(\bigvee_{i\in I} \varphi\{x/t_i\}) | I \text{ finite subset of } \mathbb{N}\} \tag{1.3}$$

Proposition 1.2.1. *With the notation of the Definition 1.2.1, the following are equivalent:*

[3] Since in Equation 1.3 there is no reference to the order used to list the closed terms of the language, it is easy to see that the definition doesn't depend on the chosen enumeration.

i) $\mathbb{P}$ *is Gaifman;*

ii)
$$\mathbb{P}(\exists x\varphi) = \lim_{n\to+\infty} \mathbb{P}(\bigvee_{i\leq n} \varphi\{x/t_i\}) \tag{1.4}$$

iii)
$$\mathbb{P}(\forall x\varphi) = \inf\{\mathbb{P}(\bigwedge_{i\in I} \varphi\{x/t_i\})|I \textit{ finite subset of } \mathbb{N}\} \tag{1.5}$$

iv)
$$\mathbb{P}(\forall x\varphi) = \lim_{n\to+\infty} \mathbb{P}(\bigwedge_{i\leq n} \varphi\{x/t_i\}) \tag{1.6}$$

Proof. Let s be the right-hand-side of Equation 1.3 and l the limit that appears in item ii) of this proposition: we will prove that $s = l$. Combining 8) and 6) of Proposition 1.0.1 we have for all $\varphi_1, \varphi_2 \in \mathrm{Sen}(\mathcal{L})$

$$\mathbb{P}(\varphi_1 \vee \varphi_2) \geq \mathbb{P}(\varphi_1),$$

and by induction we have that the succession $\{\mathbb{P}(\bigvee_{i\leq n} \varphi\{x/t_i\})\}_{n\in\mathbb{N}}$ is increasing, therefore the limit exists and

$$l := \lim_{n\to+\infty} \mathbb{P}(\bigvee_{i\leq n} \varphi\{x/t_i\}) = \sup_{n\in\mathbb{N}} \mathbb{P}(\bigvee_{i\leq n} \varphi\{x/t_i\}).$$

Since the sets of the form $\{0, 1, \ldots, n\}$ form a subset of all the finite subsets I of $\mathbb{N}$, $s \geq l$. Now, for an arbitrary ε, from the definition of sup, we have a finite $I \subset \mathbb{N}$ such that $s - \mathbb{P}(\bigvee_{i\in I} \varphi\{x/t_i\}) \leq \varepsilon$. If n is the maximum natural number in I, then $\mathbb{P}(\bigvee_{i\in I} \varphi\{x/t_i\}) \leq \mathbb{P}(\bigvee_{i\leq n} \varphi\{x/t_i\})$ and

$$s - \mathbb{P}(\bigvee_{i\leq n} \varphi\{x/t_i\}) \leq \varepsilon.$$

Therefore, as n approaches to $+\infty$ we get

$$s - l \leq \epsilon$$

and, since ε is arbitrary, $s \leq l$. Hence, $s = l$ and from this we get that *i)* is equivalent to *ii)*.

Analogously, *iii)* is equivalent to *iv)*.

Suppose now that $\mathbb{P}$ is Gaifman. Using 1) of Proposition 1.0.1, De Morgan's laws and properties of sup,

$$\begin{aligned}
\mathbb{P}(\forall x\varphi) &= \mathbb{P}(\neg\exists x\neg\varphi) = 1 - \mathbb{P}(\exists x\neg\varphi) \\
&\stackrel{\star}{=} 1 - \sup\{\mathbb{P}(\bigvee_{i\in I} \neg\varphi\{x/t_i\}) | I \text{ finite subset of } \mathbb{N}\} \\
&= 1 - \sup\{\mathbb{P}(\neg\bigwedge_{i\in I} \varphi\{x/t_i\}) | I \text{ finite subset of } \mathbb{N}\} \\
&= 1 - \sup\{1 - \mathbb{P}(\bigwedge_{i\in I} \varphi\{x/t_i\}) | I \text{ finite subset of } \mathbb{N}\} \\
&= 1 - (1 - \inf\{\mathbb{P}(\bigwedge_{i\in I} \varphi\{x/t_i\}) | I \text{ finite subset of } \mathbb{N}\}) \\
&= \inf\{\mathbb{P}(\bigwedge_{i\in I} \varphi\{x/t_i\}) | I \text{ finite subset of } \mathbb{N}\}),
\end{aligned}$$

where $\stackrel{\star}{=}$ points out where the Gaifman condition is used.

In a very similar way it can be shown that *iii)* implies the Gaifman condition, hence also *i)* and *iii)* are equivalent, from which the thesis. □

Before carrying on the study of the Gaifman property, we want to discuss a bit further its meaning. Indeed, using the previous proposition, the information about the value of $\exists x\varphi$ or of $\forall x\varphi$ depends only on the values of $\varphi\{x/t_i\}$ when t_i is a closed term. In some settings, this can be really useful, and in others annoying; we are now giving two examples.

First, let's consider having a coin that is tossed a countable number of times and you want to study some arithmetical property of the tosses sequence. Then you can introduce, in the arithmetical language $\mathcal{L}_{\text{PA}} = \{0, S, +, \cdot\}$, some "empirical symbols" (see [1]) that refers to some accidental properties of the tosses: for instance, monadic predicates $H(x)$ and $T(x)$ such that $H(n)$ is interpreted as " the n-th toss is head" and $T(n)$ as " the n-th toss is tail". If we want to describe in a more precise logical way what we are studying, we are considering the standard model of arithmetic $\mathbb{N}$ endowed with some interpretation of the empirical symbols we introduced. In this setting, the conditions in the Definition 1.0.1 make sense and also the Gaifman one: indeed, the probability we should assign to $\exists x H(x)$ should be related to the values of all the $H(n)$ where $n \in \mathbb{N}$ and here *all* these elements we are interested are interpretations of closed terms of the language (the element n in the domain $\mathbb{N}$ is the interpretation of $S^n(0)$, a closed term of PA).

However, this can be not the case in general. Maybe we have elements of a domain that can't be expressed as interpretations of closed terms: for example, if we want to study $\mathbb{R}$ with a countable language, obviously only countable elements can be described, hence, not all. Here the Gaifman condition assumes another flavor and can seem unreasonable (if we adopt a countable language).[4]

If we are interested in scenarios in which the Gaifman condition makes more sense, then we can perform the whole route of the previous subsection but dealing not with models, but instead with "separating models", i.e. models that reflect the fact that a witness of the validity of $\exists x\varphi$ is the interpretation in the model of a closed term.

Definition 1.2.2. Let $\mathfrak{A}$ be a model for a language $\mathcal{L}$. We call it *separating* if for every formula φ, with x as the only free variable,

$$\mathfrak{A} \models \exists x\varphi \text{ iff exists a closed term } t \text{ s.t. } \mathfrak{A} \models \varphi\{x/t\}.$$

From this definition more properties of separating models follow:

- assume that s and t are two terms with only the free variable x. In a model $\mathfrak{A}$, these terms can be interpreted as functions that assigns to each d element of the domain of $\mathfrak{A}$ an element of $\mathfrak{A}$: suppose that in the given model these interpretations are different. Then $\mathfrak{A}$ satisfies that $\exists x s \neq t$, hence if $\mathfrak{A}$ is separating, this distinction appears also in the domain of the interpretation of closed terms, i.e., exists a closed term r such that $t\{x/r\}$ and $s\{x/r\}$ are interpreted in different elements in $\mathfrak{A}$;

- assume that $\mathfrak{A}$ is a separating model. Then

$$\begin{aligned} \mathfrak{A} \models \forall x\varphi &\text{ iff } \mathfrak{A} \models \neg\exists x\neg\varphi \\ &\text{ iff } \mathfrak{A} \not\models \exists x\neg\varphi \\ &\text{ iff for all closed terms } t\ \mathfrak{A} \not\models \neg\varphi\{x/t\} \\ &\text{ iff for all closed terms } t\ \mathfrak{A} \models \varphi\{x/t\}. \end{aligned}$$

Let's contextualize better Definition 1.2.2 with some remarks.

Remark 1.2.1. If $\mathfrak{A}$ is a model in which all the elemnts of the domain are interpretations of closed terms of the language, then $\mathfrak{A}$ is also separating: this is the case, for instance, of the standard model of arithmetic $\mathbb{N}$ when we consider the language $\mathcal{L} = \{0, S\}$. However there are separating models in which not all the elements of the domain are interpretations of closed terms.

[4]It's not merely a problem of difference between cardinalities. Indeed, we have also countable models for $\mathcal{L}_{\mathrm{PA}}$ in which some elements can't be described as interpretations of closed terms (e.g. non-standard numbers in non-standard models of PA).

We will show an example by giving a model $\mathfrak{A}$ that is separating but not countable: since the set of all the closed terms of a countable language is countable, then there are some elements in the domain that can't be interpretations of closed terms, by cardinality issues. Consider $\mathcal{L} = \{0, S\}$, where 0 is a constant and S a function symbol and the standard model of arithmetic $\mathbb{N}$ which is an infinite $\mathcal{L}$-structure. By applying the Löwenheim-Skolem Theorem to the theory of $\mathbb{N}$, we get a model $\mathfrak{A}$ of cardinality $|\mathbb{R}|$ which is elementarily equivalent to $\mathbb{N}$. Thus, some elements in the domain of this model are not interpretations of closed terms, as explained above. However, $\mathfrak{A}$ is separating because $\mathbb{N}$ is (since all the elements of the domain are interpretations of closed terms). Indeed, if we have a sentence of the form $\exists x\varphi$ that holds in $\mathfrak{A}$, by the elementary equivalence it holds also in $\mathbb{N}$. Since $\mathbb{N}$ is separating, there is a closed term t such that $\mathbb{N} \vDash \varphi\{x/t\}$, hence by the elementary equivalence, $\varphi\{x/t\}$ holds also in $\mathfrak{A}$, from which the thesis.

Remark 1.2.2. There exist satisfiable sentences that don't have any separating model. In the language $\mathcal{L} = \{0, S\}$ [5]we can define φ as the formula $\exists x\psi$, where $\psi := (x \neq 0 \wedge \forall y\ x \neq S(y))$:

- φ is satisfiable: a model $\mathfrak{A}$ for the sentence can be defined taking $\{a, b\}$ as the domain, where 0 is interpreted as a and the function symbol S as the map that sends every element in a (this is the same model defined in Remark 1.0.1)

- no separating model $\mathfrak{B}$ satisfies φ. Indeed, if there were such a $\mathfrak{B}$, then $\mathfrak{B} \vDash \exists x\psi$ and, since $\mathfrak{B}$ is separating, it would satisfy $\psi\{x/t\}$, where t is of the form $S^n(0)$ for some $n \in \mathbb{N}$. We can distinguish two cases and every option yields a contradiction:
  - if $n = 0$, then $\mathfrak{B} \vDash \psi\{x/0\}$ but this means that $\mathfrak{B} \vDash 0 \neq 0$, a contradiction;
  - if $n \neq 0$, then $\mathfrak{B} \vDash \psi\{x/S(r)\}$, for some closed term r and this means that $\mathfrak{B} \vDash \forall y S(r) \neq S(y)$, a contradiction.

Now we can consider, among the models of the language $\mathcal{L}$, the set of the separating ones that we denote with $\widehat{\mathrm{Mod}}(\mathcal{L})$. For any statement φ, we denote $\widehat{\mathrm{Mod}}(\varphi)$ the set of separating models of φ.

[5] We are considering this language for continuity with the previous and following examples. This choice, however, may deceive the reader and not let him grasp the actual role of equality in questions about the existence of a separating model for a satisfiable formula φ. There are examples of languages also without equality that allow such kind of sentence: consider $\mathcal{L} = \{0, R\}$ where 0 is a constant and R is a relational symbol. Therefore, the only closed term is 0 and the formula $\exists x R(x) \wedge \neg R(0)$ is satisfiable but not in a separating model.

Then, with the same notations used in the case of classical models, we can define the following algebra on $\widehat{\mathrm{Mod}}(\mathcal{L})$

$$\widehat{\mathcal{B}} = \{A \subseteq \widehat{\mathrm{Mod}}(\mathcal{L}) : \exists\varphi \in \mathrm{Sen}(\mathcal{L}) \quad A = \widehat{\mathrm{Mod}}(\varphi)\}$$

and take the generated σ-algebra $\mathcal{F}(\widehat{\mathcal{B}})$. This is the same outcome that we would get if we endow $\widehat{\mathrm{Mod}}(\mathcal{L})$ with the topology induced by the basis $\widehat{\mathcal{B}}$ and then take the borelian σ-algebra. The only main difference between the setting of classical models and separating ones is that here, this topology is not compact anymore as we show in the following example.

Example 1.2.1. Let's consider the language $\mathcal{L} = \{0, S\}$ and define $\psi := x \neq S(x)$. For every n, we define the model $\mathfrak{A}_n$ having as domain the set $\{a_0, \ldots, a_n\}$, and:

- 0 is interpreted as a_0;
- $S(a_i) = a_{i+1}$ for $i = 0, \ldots, n-1$, and $S(a_n) = a_n$.

We define $C_i := \widehat{\mathrm{Mod}}(\psi(S^i(0)))$ and $D := \widehat{\mathrm{Mod}}(\forall x\psi)^c$: these sets are closed because they are complements of open sets in the topology $\widehat{\mathcal{T}}$ of $\widehat{\mathrm{Mod}}(\mathcal{L})$ ($C_i = \widehat{\mathrm{Mod}}(\neg\psi(S^i(0)))^c$).

This provides an example in which compactness fails because:

- $\cap_{i\in\mathbb{N}} C_i \cap D = \emptyset$, since there is no separating model in which all the $\psi(S^i(0))$ hold but $\forall x\psi$ doesn't; so there is no separating model that is in all the C_i's and also in D.
- for every $n \in \mathbb{N}$, $\cap_{i\leq n} C_i \cap D \neq \emptyset$, since $\mathfrak{A}_{n+1}$ is an element of this intersection. Indeed, every element of the domain of $\mathfrak{A}_{n+1}$ is an interpretation of closed terms, hence it is separating. Furthermore, it satisfies $\psi(S^i(0))$ for $i \leq n$ but $\neg\psi(S^{n+1}(0))$, hence it is in $\cap_{i\leq n} C_i$ and in $\widehat{\mathrm{Mod}}(\exists x\neg\psi) = D$.

Indeed, we can't use the same argument as in the case of classical models, and the problem relies on the fact that if a sentence has no model, it is contradictory, but this is not true anymore if the given formula has no separating models: there could be a model (non separating) that satisfies the sentence (see Remark 1.2.2).

Even if the compactness was necessary in the proof of Theorem 1.1.2, however, we have a similar result by adding the Gaifman condition. In the following by a probability on $\widehat{\mathrm{Mod}}(\mathcal{L})$, we mean a probability on $\mathcal{F}(\widehat{\mathcal{B}})$.

Theorem 1.2.1. *With the notation used before, if $\mathbb{P}$ is a Gaifman probability on $\mathcal{L}$-sentences, then there's a probability $\widehat{\mathbb{P}^*}$ on $\widehat{\mathrm{Mod}}(\mathcal{L})$ such that for every $\varphi \in \mathrm{Sen}(\mathcal{L})$,*

$$\mathbb{P}(\varphi) = \widehat{\mathbb{P}^*}(\widehat{\mathrm{Mod}}(\varphi)).$$

Conversely, let $\mathbb{P}$ be a probability on $\widehat{\mathrm{Mod}}(\mathcal{L})$. Then there is a Gaifman probability $\bar{\mathbb{P}}$ on $\mathrm{Sen}(\mathcal{L})$ such that for every $\varphi \in \mathrm{Sen}(\mathcal{L})$,

$$\bar{\mathbb{P}}(\varphi) = \mathbb{P}(\widehat{\mathrm{Mod}}(\varphi)).$$

Proof. In the following $\{t_i\}_{i\in\mathbb{N}}$ will be an enumeration of the closed terms.

For the first part of the statement, since $\mathbb{P}$ is a probability on $\mathcal{L}$-sentences, we can use the Theorem 1.1.2 in order to obtain a probability $\mathbb{P}^*$ on the $\mathrm{Mod}(\mathcal{L})$ σ-borelian algebra such that $\mathbb{P}(\varphi) = \mathbb{P}^*(\mathrm{Mod}(\varphi))$. Now we notice that $\mathrm{Mod}(\mathcal{L}) \setminus \widehat{\mathrm{Mod}}(\mathcal{L})$ is in $\mathcal{F}(\mathcal{B})$; indeed it can be described as follows:

$$\mathrm{Mod}(\mathcal{L}) \setminus \widehat{\mathrm{Mod}}(\mathcal{L}) = \bigcup_{i\in\mathbb{N}} \big(\mathrm{Mod}(\exists x\varphi_i) \cap \bigcap_{j\in\mathbb{N}} \mathrm{Mod}(\neg\varphi_i(t_j))\big),$$

where $\{\varphi_i\}_{i\in\mathbb{N}}$ an enumeration of all the $\mathcal{L}$-formulas with one and only one free variable and $\{t_i\}_{i\in\mathbb{N}}$ is an enumeration of all the closed terms. From this, it follows that $\mathcal{F}(\widehat{\mathcal{B}}) \subseteq \mathcal{F}(\mathcal{B})$ since for every $\varphi \in \mathrm{Sen}(\mathcal{L})$

$$\begin{aligned}\widehat{\mathrm{Mod}}(\varphi) &= \mathrm{Mod}(\varphi) \cap \widehat{\mathrm{Mod}}(\mathcal{L}) \\ &= \mathrm{Mod}(\varphi) \cap (\mathrm{Mod}(\mathcal{L}) \setminus \widehat{\mathrm{Mod}}(\mathcal{L}))^c.\end{aligned}$$

Therefore it makes sense to consider $\mathbb{P}^*(\mathrm{Mod}(\mathcal{L}) \setminus \widehat{\mathrm{Mod}}(\mathcal{L}))$. We can show that this value is 0. Indeed, for every i, we have

$$\mathbb{P}^*\big(\mathrm{Mod}(\exists x\varphi_i) \cap \bigcap_{j\in\mathbb{N}} \mathrm{Mod}(\neg\varphi_i(t_j))\big) = \lim_{n\to+\infty} \mathbb{P}^*\big(\mathrm{Mod}(\exists x\varphi_i) \cap \bigcap_{j\leq n} \mathrm{Mod}(\neg\varphi_i(t_j))\big)$$

Now, calling $A = \mathrm{Mod}(\exists x\varphi_i)$ and $B_n = \bigcap_{j\leq n} \mathrm{Mod}(\neg\varphi_i(t_j))$ we can note that $A^c \subseteq B_n$ because a model $\mathfrak{A}$ in which $\exists x\varphi_i$ doesn't hold, can't satisfy $\varphi_i(t_j)$

for any t_j. Therefore, using that $\mathbb{P}$ is Gaifman,

$$\begin{aligned}
\mathbb{P}^*\big(\mathrm{Mod}(\exists x\varphi_i) \cap \bigcap_{j\in\mathbb{N}} \mathrm{Mod}(\neg\varphi_i(t_j))\big) &= \lim_{n\to+\infty} \mathbb{P}^*(A \cap B_n) \\
&= \lim_{n\to+\infty} \big(\mathbb{P}^*(B_n) - \mathbb{P}^*(A^c \cap B_n)\big) \\
&= \lim_{n\to+\infty} \big(\mathbb{P}^*(B_n) - \mathbb{P}^*(A^c)\big) \\
&= \lim_{n\to+\infty} \mathbb{P}^*(B_n) - \mathbb{P}^*(A^c) \\
&= \lim_{n\to+\infty} \mathbb{P}(\bigwedge_{j\le n} \neg\varphi_i(t_j)) - \mathbb{P}(\neg\exists x\varphi_i) \\
&= \lim_{n\to+\infty} \mathbb{P}(\bigwedge_{j\le n} \neg\varphi_i(t_j)) - \mathbb{P}(\forall x\neg\varphi_i) \\
&= 0
\end{aligned}$$

Since a countable union of null measure sets has a null measure, $\mathbb{P}^*(\mathrm{Mod}(\mathcal{L})\setminus \widehat{\mathrm{Mod}}(\mathcal{L})) = 0$. Then, because $\mathcal{F}(\widehat{\mathcal{B}}) \subseteq \mathcal{F}(\mathcal{B})$, as we showed, we can restrict $\mathbb{P}^*$ to $\widehat{\mathrm{Mod}}(\mathcal{L})$ and we get $\widehat{\mathbb{P}^*}$ that is a probability as we want.

The proof of the second part of the statement is similar to the one in the Theorem 1.1.2. The only thing we should pay attention to is showing that $\bar{\mathbb{P}}$ is Gaifman. Let φ a formula with x the only free variable. Then,

$$\widehat{\mathrm{Mod}}(\exists x\varphi) = \bigcup_{i\in\mathbb{N}} \widehat{\mathrm{Mod}}(\varphi(t_i)),$$

since if $\mathfrak{A}$ is a separating model of $\exists x\varphi$, then exists a closed term t_j for which $\mathfrak{A} \vDash \varphi(t_j)$ and, conversely, if $\mathfrak{A} \vDash \varphi(t_j)$, then $\mathfrak{A} \vDash \exists x\varphi$, for any t_j. Therefore,

$$\begin{aligned}
\bar{\mathbb{P}}\,(\exists x\varphi) &= \mathbb{P}(\widehat{\mathrm{Mod}}(\exists x\varphi)) \\
&= \lim_{n\to+\infty} \mathbb{P}(\bigcup_{i\le n} \widehat{\mathrm{Mod}}(\varphi(t_i))) \\
&= \lim_{n\to+\infty} \bar{\mathbb{P}}\,(\bigvee_{i\le n} \varphi(t_i))
\end{aligned}$$

from which the thesis. $\square$

In Remark 1.2.2, we presented a sentence φ that is satisfiable but that doesn't have a separating model. This means that $\neg\varphi$ is true in all the separating models, but not valid. We next show that Theorem 1.2.1 implies that such sentences always have probability 1.

Corollary 1.2.2. *Let $\mathbb{P}$ be a Gaifman probability and φ a $\mathcal{L}$-sentence that holds in all the seprating models. Therefore, $\mathbb{P}(\varphi) = 1$.*

Proof. Since φ is true in every separating model, $\widehat{\text{Mod}}(\varphi) = \widehat{\text{Mod}}(\mathcal{L})$. By applying Theorem 1.2.1 to $\mathbb{P}$, we have a probability $\widehat{\mathbb{P}^*}$ on $\widehat{\text{Mod}}(\mathcal{L})$ such that for every sentence ψ holds that

$$\mathbb{P}(\psi) = \widehat{\mathbb{P}^*}(\widehat{\text{Mod}}(\psi)).$$

Therefore, $\mathbb{P}(\varphi) = \widehat{\mathbb{P}^*}(\widehat{\text{Mod}(\mathcal{L})})$ and, by the properties of a measure-theoretic probability, $\widehat{\mathbb{P}^*}(\widehat{\text{Mod}(\mathcal{L})}) = 1$. $\square$

1.3 Non dogmatism

The Gaifman condition is an example of a property that can't be derived by the conditions that define a probability. Another important feature that we want and that is not guaranteed by the definition of probability is non-dogmatism. A dogmatic probability is a probability that may rule out (regarding them as impossible, or more precisely, with probability zero) some statements that are satisfiable, and this can be very annoying. We formalize non-dogmatism as follows.

Definition 1.3.1. A probability $\mathbb{P}$ on $\text{Sen}(\mathcal{L})$ is called *strongly non-dogmatic* or *strongly Cournot* if for every φ that is satisfiable, $\mathbb{P}(\varphi) > 0$.

We have also a weaker notion of non-dogmatism that fits more in the Gaifman setting, i.e. in settings in which the Gaifman condition seems reasonable.

Definition 1.3.2. A probability $\mathbb{P}$ on $\text{Sen}(\mathcal{L})$ is called *non-dogmatic* or *Cournot* if for every φ that has a separating model, $\mathbb{P}(\varphi) > 0$.

It's natural to ask if there exist probabilities that meet the properties encountered: the answer to this question is given by Theorem 1.3.1, Theorem 1.3.2 and Theorem 1.3.3

Theorem 1.3.1. *Let $\mathcal{L}$ be a countable language. Then there is a probability on* $\text{Sen}(\mathcal{L})$ *that is Cournot and Gaifman.*

Proof. Let $\{\varphi_i\}_{i\in\mathbb{N}}$ be an enumeration of $\mathcal{L}$-sentences that have a separating model and let $\{\mathfrak{A}_i\}_{i\in\mathbb{N}}$ be an enumeration of some (not all) separating $\mathcal{L}$-models such that for every $i \in \mathbb{N}$, $\mathfrak{A}_i \vDash \varphi_i$.

We can consider then the mass function $m : \mathbb{N} \to [0,1]$ that assigns to every $i \in \mathbb{N}$ the value $\frac{1}{(i+1)(i+2)}$[6]. Since $m(i) \geq 0$ and $\sum_{i\in\mathbb{N}} m(i) = 1$, this determines a probability $\mathbb{P}_m$ (in the measure-theoretical sense) on $\mathrm{Mod}(\mathcal{L})$ that assigns to every $A \in \mathcal{F}(\mathcal{B})$ the value

$$\mathbb{P}_m(A) = \sum_{\{i:\mathfrak{A}_i \in A\}} m(i).$$

By Theorem 1.1.2, $\mathbb{P}_m$ induces a probability $\bar{\mathbb{P}}_m$ on $\mathrm{Sen}(\mathcal{L})$; this is Cournot since every formula φ that has a separating model is a certain φ_j of the list that appears at the beginning of the proof and therefore $\bar{\mathbb{P}}_m(\varphi) \geq \frac{1}{(j+1)(j+2)}$. Furthermore, $\bar{\mathbb{P}}_m$ is also Gaifman, because according to the probability $\mathbb{P}_m$, the set $\mathrm{Mod}(\mathcal{L}) \setminus \widehat{\mathrm{Mod}}(\mathcal{L})$ has null measure and, therefore, restricting $\mathbb{P}_m$ to a probability on $\widehat{\mathrm{Mod}}(\mathcal{L})$, we get, by Theorem 1.2.1, that $\bar{\mathbb{P}}_m$ is Gaifman. $\square$

Before going on to discuss the existence of probabilities that have the properties encountered before, we present a nice characterization of Gaifman and Cournot probabilities.

Definition 1.3.3. Let $\{\varphi_i\}_{i\in\mathbb{N}}$ be an enumeration without repetition of all sentences that have a separating model. A probability $\mathbb{P}$ on $\mathrm{Sen}(\mathcal{L})$ has a *rigid mixture representation*[7] if there exist probabilities $\{\mathbb{P}_i\}_{i\in\mathbb{N}}$ such that:

- $\mathbb{P} = \sum_{i=0}^{+\infty} m_i \mathbb{P}_i$, where m_i are positive real numbers that sum up to 1 (i.e. $\sum_{i=0}^{+\infty} m_i = 1$);
- for every $i \in \mathbb{N}$, $\mathbb{P}_i(\varphi_i) = 1$.

Proposition 1.3.1. *Let $\mathbb{P}$ be a probability on* $\mathrm{Sen}(\mathcal{L})$. *Then $\mathbb{P}$ is Cournot and Gaifman if and only if it has a rigid mixture representation and the $\mathbb{P}_i$ of the representation (if we use the notation of Definition 1.3.3) are Gaifman.*

Proof. Let $\{\varphi_i\}_{i\in\mathbb{N}}$ be an enumeration without repetition of sentences that are satisfiable in a separating model.

First we show that if a probability $\mathbb{P}$ has a rigid mixture representation $\sum_{i=0}^{+\infty} m_i \mathbb{P}_i$, it is Cournot. Assume that ψ has a separating model: then, $\psi = \varphi_i$ for some $i \in \mathbb{N}$ and $\mathbb{P}_i(\psi) = 1$. From $m_i > 0$ follows that

$$\mathbb{P}(\psi) \geq m_i \mathbb{P}_i(\psi) = m_i > 0.$$

[6] Actually we can choose also other values for $m(i)$; the only properties we require for the proof is that they are positive values and $\sum_{i\in\mathbb{N}} m(i) = 1$

[7] A careful inspection of this definition points out that the property of having a rigid mixture representation depends on the enumeration $\{\varphi_i\}_{i\in\mathbb{N}}$. It's easy to show that the definition is not ambiguous, since a probability that has a rigid mixture representation for a given enumeration, satisfies the property with any other enumeration.

If the $\mathbb{P}_i$'s are Gaifman, then also $\mathbb{P}$ is so, since a linear combination of Gaifman is Gaifman. Indeed, for every formula φ with no free variables but x and an enumeration of the closed terms $\{t_i\}_{i\in\mathbb{N}}$, we have

$$\begin{aligned}
\mathbb{P}(\exists x\varphi) &= \sum_{i=0}^{+\infty} m_i\,\mathbb{P}_i(\exists x\varphi)\\
&= \sum_{i=0}^{+\infty} m_i(\lim_{n\to+\infty}\mathbb{P}_i(\bigvee_{j\leq n}\varphi\{x/t_j\}))\\
&= \lim_{n\to+\infty}(\sum_{i=0}^{+\infty} m_i\,\mathbb{P}_i(\bigvee_{j\leq n}\varphi\{x/t_j\}))\\
&= \lim_{n\to+\infty}\mathbb{P}(\bigvee_{j\leq n}\varphi\{x/t_j\}),
\end{aligned}$$

where the possibility of switching the limit and the summation is guaranteed by the Monotone Convergence Theorem (for every $i\in\mathbb{N}$, the succession $\{m_i\,\mathbb{P}_i(\bigvee_{j\leq n}\varphi\{x/t_j\})\}_{n\in\mathbb{N}}$ is increasing as n becomes bigger).

Now, assume that we have a Gaifman and Cournot probability $\mathbb{P}$ and we will find a rigid mixture representation for it. Considering the enumeration $\{\varphi_i\}_{i\in\mathbb{N}}$ of the sentences that have a separating model, let:

$$\begin{aligned}
\mathcal{E} := \{i\in\mathbb{N}\ :\ &\varphi_i \text{ starts with an even (zero included) number of } \neg \text{ and}\\
&\neg\varphi_i \text{ is satisfiable in a separating model}\}\\
\mathcal{O} := \{i\in\mathbb{N}\ :\ &\varphi_i \text{ starts with an odd number of } \neg \text{ and}\\
&\neg\varphi_i \text{ is satisfiable in a separating model}\}.
\end{aligned}$$

To every $i\in\mathcal{E}$ we can associate a unique $c(i)\in\mathcal{O}$ such that $\varphi_{c(i)}=\neg\varphi_i$.

Indeed, if $i\in\mathcal{E}$, $\neg\varphi_i$ starts with an odd number of $\neg$ and it is satisfiable in a separating model, so it exists an index j ($=c(i)$, unique because we don't have repetitions in the enumeration) such that $\varphi_j=\neg\varphi_i$. This function c is bijective since:

- it is injective because the enumeration is assumed to be without repetition;
- it is surjective, because if $j\in\mathcal{O}$, then exists ψ such that $\varphi_j=\neg\psi$ and ψ starts with an even number of $\neg$. The formula ψ is satisfiable in a separating model because, for being $j\in\mathcal{O}$, $\neg\varphi_j$ is satisfiable in a separating model and $\neg\varphi_j\equiv\psi$. Hence, there exists $i\in\mathbb{N}$ such that $\varphi_i=\psi$ and $\neg\varphi_i$ is equivalent to φ_j, so $\neg\varphi_i$ is satisfiable in a separating model. Therefore $i\in\mathcal{E}$ and $c(i)=j$.

If an index i doesn't belong to $\mathcal{E} \cup \mathcal{O}$, it means that $\neg\varphi_i$ is not satisfiable in a separating model, therefore φ_i holds in all separating models. We can define then

$$\mathcal{T} := \{i \in \mathbb{N} \ : \ \varphi_i \text{ holds in all separating models}\}$$

and we get in this way a partition of $\mathbb{N}$ in the subsets $\mathcal{E}, \mathcal{O}$ and $\mathcal{T}$.

We now define $\mathbb{P}_i$ in different ways, according to the subset in which the index i belongs:

- if $i \in \mathcal{E}$, since $\varphi_i \vee \neg\varphi_i$ holds in all models, then for any formula ψ in $\mathrm{Sen}(\mathcal{L})$, we have

$$\begin{aligned}
\mathbb{P}(\psi) &= \mathbb{P}(\psi \wedge (\varphi_i \vee \neg\varphi_i)) \\
&= \mathbb{P}(\psi \wedge \varphi_i) + \mathbb{P}(\psi \wedge \neg\varphi_i) \\
&= \mathbb{P}(\psi|\varphi_i)\,\mathbb{P}(\varphi_i) + \mathbb{P}(\psi|\neg\varphi_i)\,\mathbb{P}(\neg\varphi_i) \\
&= \mathbb{P}(\psi|\varphi_i)\,\mathbb{P}(\varphi_i) + \mathbb{P}(\psi|\varphi_{c(i)})\,\mathbb{P}(\varphi_{c(i)}).
\end{aligned}$$

  We can define $\mathbb{P}_i := \mathbb{P}(-|\varphi_i)$ and $\mathbb{P}_{c(i)} := \mathbb{P}(-|\varphi_{c(i)})$: in this way, for all $\psi \in \mathrm{Sen}(\mathcal{L})$ it holds

$$\mathbb{P}(\psi) = \mathbb{P}(\varphi_i)\,\mathbb{P}_i(\psi) + \mathbb{P}(\varphi_{c(i)})\,\mathbb{P}_{c(i)}(\psi).$$

  Since c is bijective, this is a definition not only for the indices in $\mathcal{E}$ but also for the ones in $\mathcal{O}$. Thus, denoting p_j the value $\mathbb{P}(\varphi_j)$ for $j \in \mathcal{E} \cup \mathcal{O}$, we have that, if $i \in \mathcal{E}$, then

$$\mathbb{P} = p_i\,\mathbb{P}_i + p_{c(i)}\,\mathbb{P}_{c(i)}\,.$$

- if $i \in \mathcal{T}$, then φ_i holds in every separating model. Since $\mathbb{P}$ is Gaifman, this means that $\mathbb{P}(\varphi_i) = 1$, by Corollary 1.2.2. Thus, for all $\psi \in \mathrm{Sen}(\mathcal{L})$ it holds

$$\begin{aligned}
\mathbb{P}(\psi) &= \mathbb{P}(\psi \wedge (\varphi_i \vee \neg\varphi_i)) \\
&= \mathbb{P}(\psi \wedge \varphi_i) + \mathbb{P}(\psi \wedge \neg\varphi_i) \\
&\overset{*}{=} \mathbb{P}(\psi|\varphi_i)\,\mathbb{P}(\varphi_i) \\
&= \mathbb{P}(\psi|\varphi_i),
\end{aligned}$$

  where in $\overset{*}{=}$ we use that $\mathbb{P}(\neg\varphi_i) = 0$ and, therefore, that $\mathbb{P}(\psi \wedge \neg\varphi_i) = 0$. Defining $\mathbb{P}_i := \mathbb{P}(-|\varphi_i)$ also in this case, we have for every $i \in \mathcal{T}$ that $\mathbb{P} = \mathbb{P}_i$.

Consider now a function $r : \mathcal{E} \to [0,1]$ such that $\sum_{i\in\mathcal{E}} r(i) = 1$. Then, for all $\psi \in \mathrm{Sen}(\mathcal{L})$ we have

$$\begin{aligned} \mathbb{P}(\psi) &= \sum_{i\in\mathcal{E}} r(i)\,\mathbb{P}(\psi) \\ &= \sum_{i\in\mathcal{E}} r(i)(p_i\,\mathbb{P}_i(\psi) + p_{c(i)}\,\mathbb{P}_{c(i)}(\psi)) \\ &= \sum_{j\in\mathcal{E}\dot\cup\mathcal{O}} 2m_j\,\mathbb{P}_j(\psi), \end{aligned} \tag{1.7}$$

where $j \in \mathcal{E}$, $m_j = \frac{1}{2}r(j)p_j$, while for $j \in \mathcal{O}$, $m_j = \frac{1}{2}r(c^{-1}(j))p_{c(j)}$.

In the same way, if we choose some $m_i \in [0,1]$ for every $i \in \mathcal{T}$, such that $\sum_{i\in\mathcal{T}} m_i = \frac{1}{2}$, we get

$$\mathbb{P}(\psi) = \sum_{i\in\mathcal{T}} 2m_i\,\mathbb{P}_i(\psi). \tag{1.8}$$

Using Equation (1.7) and (1.8), we have for a generic ψ

$$\begin{aligned} \mathbb{P}(\psi) &= \frac{1}{2}\,\mathbb{P}(\psi) + \frac{1}{2}\,\mathbb{P}(\psi) \\ &= \frac{1}{2}\sum_{i\in\mathcal{T}} 2m_i\,\mathbb{P}_i(\psi) + \frac{1}{2}\sum_{j\in\mathcal{E}\dot\cup\mathcal{O}} 2m_j\,\mathbb{P}_j(\psi) \\ &= \sum_{i\in\mathcal{T}} m_i\,\mathbb{P}_i(\psi) + \sum_{j\in\mathcal{E}\dot\cup\mathcal{O}} m_j\,\mathbb{P}_j(\psi) \\ &= \sum_{k\in\mathbb{N}} m_k\,\mathbb{P}_k(\psi). \end{aligned}$$

Hence, we have the thesis noticing that:

- for every $i \in \mathbb{N}$, $\mathbb{P}_i(\varphi_i) = \mathbb{P}(\varphi_i|\varphi_i) = 1$;
- $\mathbb{P}_i$ is Gaifman for every $i \in \mathbb{N}$. This is a fact that holds for a generic sentence ψ (in which, therefore, there is no free variables): if $\mathbb{P}$ is Gaifman and $\mathbb{P}(\psi) > 0$, then $\mathbb{P}(-|\psi)$ is Gaifman. Indeed, for a sentence

$\exists x\varphi$ and for an enumeration of closed terms $\{t_i\}_{i\in\mathbb{N}}$, we get

$$\begin{aligned}\mathbb{P}(\exists x\varphi|\psi) &= \frac{\mathbb{P}(\exists x\varphi \wedge \psi)}{\mathbb{P}(\psi)}\\ &= \frac{\mathbb{P}(\exists x(\varphi \wedge \psi))}{\mathbb{P}(\psi)}\\ &= \frac{\lim_{n\to+\infty}\mathbb{P}(\bigvee_{j\leq n}(\varphi\wedge\psi)\{x/t_j\})}{\mathbb{P}(\psi)}\\ &= \frac{\lim_{n\to+\infty}\mathbb{P}(\bigvee_{j\leq n}\varphi\{x/t_j\}\wedge\psi)}{\mathbb{P}(\psi)}\\ &= \lim_{n\to+\infty}\frac{\mathbb{P}(\bigvee_{j\leq n}\varphi\{x/t_j\}\wedge\psi)}{\mathbb{P}(\psi)}\\ &= \lim_{n\to+\infty}\mathbb{P}(\bigvee_{j\leq n}\varphi\{x/t_j\}|\psi).\end{aligned}$$

□

Theorem 1.3.2. *Let $\mathcal{L}$ be a countable language. Then there is a probability on* $\mathrm{Sen}(\mathcal{L})$ *that is strongly Cournot.*

Proof. Let $\{\varphi_i\}_{i\in\mathbb{N}}$ be an enumeration of $\mathcal{L}$-sentences that have a model and let $\{\mathfrak{A}_i\}_{i\in\mathbb{N}}$ be an enumeration of $\mathcal{L}$-models such that for every $i\in\mathbb{N}$, $\mathfrak{A}_i \vDash \varphi_i$.

As in the Theorem 1.3.1, we can consider the mass function $m:\mathbb{N}\to[0,1]$ that assigns to every i the value $\frac{1}{(i+1)(i+2)}$. It determines a probability $\mathbb{P}_m$ on $\mathrm{Mod}(\mathcal{L})$. By Theorem 1.1.2, $\mathbb{P}_m$ induces a probability $\bar{\mathbb{P}}_m$ on $\mathrm{Sen}(\mathcal{L})$; this is strongly Cournot since every satisfiable formula φ is a certain φ_j of the list that appears at the beginning of the proof and therefore $\bar{\mathbb{P}}_m(\varphi) = \frac{1}{(j+1)(j+2)}$. □

If we are in a context in which the Gaifman condition seems reasonable, the right notion of non-dogmatism appears to be the Cournot one. However, it is still interesting that we can't have in some settings a probability that is both strongly Cournot and Gaifman.

Theorem 1.3.3. *There is no strongly Cournot and Gaifman probability on the language $\mathcal{L}=\{0,S\}$, with 0 constant and S function symbol.*

Proof. In Remark 1.2.2 we provided a formula $\psi(x) := (x\neq 0 \wedge \forall y x\neq S(y))$ such that for every $n\in\mathbb{N}$, $\bigvee_{i\leq n}\psi(S^i(0))$ is unsatisfiable even if $\exists x\psi(x)$ has a model.

Therefore, since every closed term t is of the form $S^i(0)$ for some $i \in \mathbb{N}$, for every enumeration $\{t_i\}_{i\in\mathbb{N}}$ of closed terms of the language and for every $n \in \mathbb{N}$, $\bigvee_{i\leq n} \psi(t_i)$ is unsatisfiable.

If $\mathbb{P}$ is strongly Cournot and Gaifman on $\mathrm{Sen}(\mathcal{L})$, we have:

- for being strongly Cournot, $\mathbb{P}(\exists x\psi(x)) > 0$ because the sentence is satisfiable;
- for being a probability for all $n \in \mathbb{N}$, $\mathbb{P}(\bigvee_{i\leq n} \psi(t_i)) = 0$ because the sentence $\bigvee_{i\leq n} \psi(t_i)$ is unsatisfiable;
- for being Gaifman,

$$\mathbb{P}(\exists x\psi(x)) = \lim_{n\to+\infty} \mathbb{P}(\bigvee_{i\leq n} \psi(t_i)).$$

These three conditions yield a contradiction. □

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